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THE EFFECT OF WALL FRICTION ON THE STRENGTH OF SHOCK WAVES IN TUBES AND HYDRAULIC JUMPS IN CHANNELS

By Coleman duP. Donaldson and Roger D. Sullivan

SUMMARY

A theory is presented for the attenuation due to viscous boundary layer of shock waves traveling in closed passages. Because of the similar nature of the problem, the theory is also developed for the attenuation of a hydraulic jump moving in a channel. The results of experiments that bear out the theory are presented for both a shock tube and a hydraulic channel.

INTRODUCTION

Recent interest in intermittent-flow devices, as, for example, shock tubes, has caused questions to be raised concerning the viscous effects.

Although a large amount of literature exists on the theory of shock tubes in which perfect fluids are used (see, for example, references 1 and 2), no discussion is known to be available of the effects of viscosity in such devices. The present analysis is an attempt to develop a quantitative method for determining the attenuation of a shock wave moving in a tube or passage resulting from the action of viscosity at the walls.

Because of the similar nature of the attenuation of a hydraulic jump traveling in a channel, the method is also developed for this phenomenon.

SYMBOLS

a velocity of small disturbances in fluid

A area of tube

g acceleration due to gravity

h depth of liquid

H depth ratio (h_2/h_1)

$l(H) = (H - 1)\sqrt{H(H + 1)}$

m mass flow per unit time

M Mach number

$n(P) = \sqrt{6P + 1}\,\frac{P - 1}{P + 6}$

p pressure

P pressure ratio (p_2/p_1)

r radial coordinate from center of tube

R radius of cylindrical tube

t time

T absolute temperature

u velocity

w width of channel

x horizontal coordinate

y vertical coordinate

γ ratio of specific heats

μ absolute viscosity

ν kinematic viscosity (μ/ρ)

ρ density

Subscripts:

0 conditions in fluid on high-pressure side of diaphragm or barrier

1 conditions in fluid on low-pressure side of diaphragm or barrier

2 conditions in fluid behind shock or hydraulic jump

3 conditions in fluid after wave reflection

ex pertaining to expansion wave

i pertaining to initial flow conditions

s pertaining to shock or hydraulic jump

BOUNDARY LAYER IN A TUBE IN WHICH THE ENTIRE FLUID IS SUDDENLY ACCELERATED

Since the boundary layers that cause the attenuation considered are assumed to depend principally on time, the differential equation of viscous motion is solved for the case of an infinitely long tube of radius R in which the entire fluid at time $t = 0$ is suddenly set in motion at velocity u_i. In this case, the velocities caused by viscous action depend only upon the distance from the center of the tube and on the time that the fluid has been in motion, rather than upon position along the length of the tube; therefore, the differential equation of viscous motion in cylindrical coordinates becomes

$$\frac{\partial u}{\partial t} = \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) \tag{1}$$

The boundary conditions to be imposed are:

at $t = 0$

$$\left.\begin{aligned} &u = u_i && (0 \leqq r < R) \\ &\text{and at } r = R \\ &u = 0 && (t > 0) \end{aligned}\right\} \tag{2}$$

In addition to these boundary conditions, the function $u = f(r,t)$ must be continuous at $r = 0$.

The solution of this equation subject to these boundary conditions is well-known in the theory of heat conduction (see reference 3) and is

$$\frac{u}{u_1} = 2\sum_{n=1}^{\infty} \frac{J_0\left(\frac{\alpha_n r}{R}\right)e^{-\alpha_n^2 \nu t/R^2}}{\alpha_n J_1(\alpha_n)} \tag{3}$$

where the values of α_n are the roots of the equation $J_0(x) = 0$ and J_0 and J_1 are Bessel functions of the first kind of order 0 and 1, respectively.

The velocity profiles in the tube for various values of the time parameter $\nu t/R^2$ are plotted in figure 1. This solution for small values of the time parameter reduces to the two-dimensional solution

$$\frac{u}{u_1} = \operatorname{erf} \frac{y}{2\sqrt{\nu t}}$$

where

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2}\, dx$$

and y is the distance from the surface at which the velocity is 0 and in this case is equal to $R - r$. This solution is useful not only for evaluating the boundary layer in tubes of other than circular cross section but also for evaluating the series of equation (3) for small values of the time parameter since, for these values of the parameter, many terms of the series are necessary.

Figure 1 indicates that, as time passes, the mass flow in the tube decreases from its maximum and initial value at $t = 0$ of $m_i = \pi\rho u_i R^2$.

The ratio of the mass flow at any time to the initial mass flow

$$\frac{m}{m_i} = \frac{\int_0^R 2\pi\rho r u\, dr}{\pi\rho u_i R^2} = 4\sum_{n=1}^{\infty} \frac{1}{\alpha_n^2}\, e^{-\alpha_n^2 \nu t/R^2} \tag{4}$$

is plotted in figure 2 as a function of the time parameter $\nu t/R^2$.

For small values of the time parameter equation (4) reduces to the two-dimensional solution

$$\frac{m}{m_1} = 1 - 4\sqrt{\frac{\nu t}{A}}$$

where A is the area of the tube, so that for the circular tube

$$\frac{m}{m_1} = 1 - \frac{4}{\sqrt{\pi}}\sqrt{\frac{\nu t}{R^2}}$$

This result is also plotted in figure 2 for comparison with the results of equation (4). These solutions are used to discuss the attenuation of a shock wave traveling in a tube.

ATTENUATION OF A SHOCK WAVE IN A TUBE

A shock tube is a closed tube that is divided into two compartments by means of a breakable diaphragm. If the two compartments are pumped to different pressures p_0 and p_1 where $p_0 > p_1$ and the diaphragm is broken, an expansion wave travels into the compartment where the air is at rest at p_0; the air is thus accelerated to velocity u_2 and the pressure decreased from p_0 to p_2. A shock wave travels into the compartment where the air is at rest at pressure p_1, accelerates it to the same velocity u_2, and raises its pressure to the same pressure p_2. This process is illustrated in figure 3. The result of bursting the diaphragm in the absence of viscosity is to set the entire mass of air between the expansion wave and the shock wave in motion at velocity u_2. Thus, the mass flow per unit time set in motion across a section of the tube between the diaphragm and the shock wave is

$$m = \pi\rho_2 u_2 R^2$$

This mass flow is related to the pressure ratio across the shock wave $\frac{p_2}{p_1} = P$ and the conditions ahead of the wave by the following equation (for $\gamma = 1.4$)

$$\frac{m}{\pi\rho_1 a_1 R^2} = \frac{5}{\sqrt{7}}\sqrt{6P + 1}\,\frac{P - 1}{P + 6} = \frac{5}{\sqrt{7}}\,n(P) \tag{5}$$

which is derived in the appendix; thus,

$$n(P) = \sqrt{6P + 1}\,\frac{P - 1}{P + 6} \tag{6}$$

In order to evaluate the effect of viscosity upon the flow in the tube, the following assumption is made: The thickness of the boundary layer at any point behind the shock depends only upon the length of time the air has been in motion with respect to the wall of the tube, or upon the length of time since the shock wave passed those particles. Such an assumption neglects variations of velocity in the stream direction and can be shown to give an answer to the steady flat-plate boundary-layer flow that is a good approximation to the Blasius solution. Since, in the present problem, the shock wave travels faster than the fluid in motion behind it, this assumption should be an even better approximation to the real solution. Thus, the boundary layer between the shock wave and the particles that were originally at the diaphragm is essentially as shown in figure 4.

When this assumption is made, the solution obtained in the preceding section is applicable. If attention is confined to the position in the tube at which the particles originally at the diaphragm are found, the mass flow in this section must drop off with time according to equation (4), so that the mass flow at any time t is

$$m = m_1\, 4\sum_{n=1}^{\infty} \frac{1}{\alpha_n^2}\, e^{-\alpha_n^2 \nu t/R^2}$$

If the value of the initial mass flow in terms of the initial pressure ratio (equation (5)) is substituted in this expression, the mass flow at any time is given by

$$\frac{m}{\pi\rho_1 a_1 R^2} = \frac{5}{\sqrt{7}}\sqrt{6P_i + 1}\,\frac{P_i - 1}{P_i + 6}\,4\sum_{n=1}^{\infty}\frac{1}{\alpha_n^2}\,e^{-\alpha_n^2 \nu t/R^2} \tag{7}$$

This formula indicates that the mass flow in the tube decreases with time according to the function that is plotted in figure 2.

Furthermore, the shock wave itself sets in motion a mass flow that is given by (see equation (5))

$$\frac{m}{\pi\rho_1 a_1 R^2} = \frac{5}{\sqrt{7}}\sqrt{6P + 1}\,\frac{P - 1}{P + 6}$$

If this mass flow is greater than that crossing the tube at the position of the diaphragm particles, an expansion wave would be formed that would catch up with the shock wave, reduce its strength, and thus reduce the mass flow it was initiating; but if the mass flow caused by the shock was less than that across the diaphragm-particle section, a compression wave would develop that would move after the wave, catch it, increase its strength, and thus increase the mass flow it was initiating. Hence, it may be assumed for the purposes of estimating the attenuation that the mass flow initiated by the wave is just equal to that given by equation (7). If both the expansion wave and the compression wave were reduced in this manner, a pressure gradient would be produced between these two waves in order to preserve the pressure ratio between the two undisturbed regions. Part of this pressure gradient would be used to accelerate somewhat the flow between the two waves, and thus the decrease in mass flow at the diaphragm-particle section would be somewhat less than that given by equation (7). However, for the purpose of determining the approximate magnitude of the attenuation, the effect of this pressure gradient is neglected. If the mass flows given by equations (5) and (7) are set equal to each other, the relationship between the strength of the wave P and its initial strength P_i is found to be

$$\sqrt{6P + 1}\,\frac{P - 1}{P + 6} = \sqrt{6P_i + 1}\,\frac{P_i - 1}{P_i + 6}\,4\sum_{n=1}^{\infty}\frac{1}{\alpha_n^2}\,e^{-\alpha_n^2 \nu t/R^2} \tag{8}$$

The function

$$n(P) = \sqrt{6P + 1}\,\frac{P - 1}{P + 6}$$

is plotted in figure 5 as a function of the strength of the wave P. Substituting $n(P)$ from equation (6) in equation (8) gives

$$\frac{n(P)}{n(P_1)} = 4 \sum_{n=1}^{\infty} \frac{1}{\alpha_n^2} e^{-\alpha_n^2 \nu t/R^2} \tag{9}$$

which is the same function as is plotted in figure 2.

If the strength of the shock at any distance from the diaphragm is to be determined, the values of t need only be found in terms of the distance x and the velocity of the shock u_s and the kinematic viscosity behind the shock evaluated in terms of the shock strength P and the conditions ahead of the shock. If the attenuation is small,

$$t \approx \frac{x}{u_{s_1}}$$

Then, since u_{s_1} is a function of the strength of the shock and the condition ahead of the shock, $\nu_2 t/R^2$ becomes (see derivation in the appendix)

$$\frac{\nu_2 t}{R^2} = \frac{\sqrt{7}(P_1 + 6)^2 P_1}{(6P_1 + 1)^{2.5}} \frac{\nu_1 x}{a_1 R^2} \tag{10}$$

where linear variation of viscosity with temperature is assumed.

If the initial strength of the wave and the conditions into which it is traveling are known, $\nu_2 t/R^2$ can be evaluated for any distance from the diaphragm. After this value is determined, the ratio $n(P)/n(P_1)$ may be found from figure 2. Then, from figure 5, the value of the strength of the wave P at x may be evaluated.

Before proceeding with an account of an experimental test of this theory, the theory for the attenuation of a hydraulic jump in a channel is presented.

BOUNDARY LAYER IN A FLUID OF FINITE DEPTH

An analysis parallel to that of the boundary layer in a cylinder may be carried out for the boundary layer in a fluid of finite depth on a flat plate. This analysis can be applied to find the attenuation of a hydraulic jump traveling in a channel.

The differential equation (corresponding to equation (1)) is

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}$$

The boundary conditions are:

at $t = 0$

$$u = u_i \qquad (0 < y \leqq h)$$

at $y = 0$

$$u = 0 \qquad (t > 0)$$

at $y = h$

$$\frac{\partial u}{\partial y} = 0 \qquad (t > 0)$$

The last condition expresses the absence of tangential force on the free surface.

The solution of this boundary-value problem is also known from the theory of heat transfer. (See reference 3.) The solution is

$$\frac{u}{u_i} = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n - 1}\sin\left[(2n - 1)\frac{\pi y}{2h}\right]e^{-\frac{\pi^2}{4}(2n-1)^2\frac{\nu t}{h^2}}$$

The ratio of the mass flow past a section of width w at any time t

$$m = w\rho\int_0^h u\, dy$$

to its value at $t = 0$ of

$$m_i = w\rho h u_i$$

is

$$\frac{m}{m_i} = \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n - 1)^2}\, e^{-\frac{\pi^2}{4}(2n-1)^2\frac{\nu t}{h^2}} \tag{11}$$

This result is plotted in figure 6 and is used to evaluate the attenuation of a hydraulic jump traveling in a channel.

ATTENUATION OF A HYDRAULIC JUMP IN A CHANNEL

If a channel is divided into two parts by a barrier and the two parts are filled with a liquid to different levels h_0 and h_1, where $h_0 > h_1$, and the barrier is suddenly removed, a situation analogous to that of a shock tube exists. Although the equations of a shock and a hydraulic jump are completely different (not just the same equation using different ratios of specific heats, as in the case of continuous flows such as the water analogy, see reference 4) the phenomena are fundamentally similar; in particular, the liquid originally at a depth h_1 is suddenly accelerated to a velocity u_2 and the depth is increased to h_2 when the jump passes by it.

The mass flow per unit time set in motion immediately behind a hydraulic jump in a channel of width w is

$$m = \rho w u_2 h_2$$

This mass flow is related to the ratio of the depths on the two sides of the jump $\frac{h_2}{h_1} = H$ and to the conditions ahead of the wave by the equation (derived in the appendix):

$$\frac{m}{\rho w a_1 h_1} = \frac{1}{\sqrt{2}}\,(H - 1)\sqrt{H(H + 1)} = \frac{1}{\sqrt{2}}\,l(H) \tag{12}$$

where $a_1 = \sqrt{gh_1}$ is the speed of an infinitesimal wave. The parameter $l(H)$ is plotted as a function of the depth ratio H in figure 7.

By an argument parallel to that for a shock tube, if the channel is assumed to be wide in comparison with the depth of the liquid so that edge effects may be neglected, the strength of the wave after it has gone a distance x can be determined by the relation

$$\frac{l(H)}{l(H_1)} = \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} e^{-\frac{\pi^2}{4}(2n-1)^2\frac{\nu t}{h_2^2}} \tag{13}$$

If $\nu t/h_2^2$ is expressed in terms of the jump strength and conditions ahead of the wave, and if the assumption is made that $t = \frac{x}{u_{s_1}}$, this parameter becomes, as shown in the appendix:

$$\frac{\nu t}{h_2^2} = \frac{1}{H^2}\sqrt{\frac{2}{H(H-1)}}\frac{\nu x}{a_1 h_1^2} \tag{14}$$

With the aid of equation (14) and figures 6 and 7, the strength of a hydraulic jump at a position in the channel can be computed.

TESTS OF THE ATTENUATION OF A SHOCK WAVE IN A TUBE

In order to obtain an experimental check of the theory just presented, a special shock tube was constructed. The tube was 0.4 inch in diameter, and the high-pressure compartment was $4\frac{1}{2}$ feet from the diaphragm to the end of the tube. Two low-pressure chambers were interchangeable on the other side of the diaphragm – the first 1 foot long, the other 11.3 feet long from the diaphragm to the pickup crystal. The experimental apparatus with the short low-pressure chamber is shown in figures 8 and 9.

Tests were made by charging the high-pressure chamber with nitrogen to the desired pressure. The diaphragm was then broken by puncturing it with a solenoid-driven plunger, and a shock wave passed into the

low-pressure chamber, which was at atmospheric pressure, reached the end of the tube, and was reflected back toward the diaphragm. The voltage output of a piezoelectric crystal mounted flush in the end of the tube was recorded. This voltage, which was a measure of the ratio of the initial pressure in the tube and the pressure after the shock wave had been reflected, was recorded by means of an oscillograph and rotating drum camera the same as that described in reference 2. One such record, which is typical, is presented herein as figure 10.

By this method the voltage output of the crystal was obtained for a range of initial ratios of pressure between the high-pressure and low-pressure sides of the diaphragm with both the 1-foot and the 11.3-foot pressure chambers. Since the reflected-pressure ratio measured (p_3/p_1) is related to the strength of the wave striking the end of the tube by the formula (see the appendix)

$$\frac{p_3}{p_1} = \frac{P(8P - 1)}{P + 6}$$

the records made are a measure of the relative attenuation of the wave at stations 1 foot and 11.3 feet from the diaphragm and are seen to be independent of the crystal characteristics.

The theoretical reflected-pressure ratio p_3/p_1 as a function of initial-pressure ratio p_0/p_1, with and without the effect of viscosity considered, is plotted in figure 11 for a station 1 foot from the diaphragm, along with the voltage output of the crystal measured at the same station. The derivation of the relation between p_0/p_1 and P is given in the appendix.

If the theory is assumed to be correct, these data may be used to determine the voltage output of the crystal for a given pressure rise. This information is given in figure 12. The reflected-pressure ratios corresponding to the crystal voltage outputs measured with the 11.3-foot tube can be determined from this figure and are plotted in figure 13, along with the theoretical pressure ratio at this station.

Although the theory seems always to predict too little attenuation, the order of magnitude (12 to 15 percent) agrees very well with that measured (12.5 to 16.7 percent). However, the theory takes into account only one factor contributing to the attenuation, and other factors (such as the impossibility of a perfectly efficient diaphragm) may contribute to a reduction of the strength of the initial wave formed which might well be the order of the difference between the theory and experiment.

TESTS OF THE ATTENUATION OF A HYDRAULIC JUMP IN A CHANNEL

For experimentally determining the attenuation of hydraulic jumps an apparatus was used that consisted of a standard 8-inch steel channel 20 feet long with plates welded to its ends. A transverse cut was made in the middle, into which a removable metal barrier was inserted. Liquid was poured in to different depths on the two sides of the barrier. The barrier was suddenly removed, and the depth just behind the wave was measured at two points – one close to the barrier, and one farther downstream. The depths were measured in sixty-fourths of an inch by means of thin steel scales. By this means the height of the moving liquid could be obtained within 3 percent.

The first tests were made with water. The attenuation of the waves was found to be about twice that predicted by the theory. However, surface tension, unaccounted for in the theory, was believed to contribute a large share of the attenuation. In order to avoid this effect, SAE 10 oil, which has a surface tension about one-third that of water and a kinematic viscosity about seventy times that of water, was used so that the effect of surface tension would be negligible compared to that of the viscous boundary layer. Measurements were made of the attenuation of hydraulic jumps of various strengths moving into liquid depths of from 30/64 inch to 60/64 inch. The height measurements for this purpose were made at a position 2 inches downstream and at a position 4 feet downstream from the barrier.

Some of the results are given in figure 14. Since the derivation of the relation between H_1 and h_0/h_1 is straightforward (see appendix) the experimental values of the hydraulic jump strength H measured close to the barrier position plotted against h_0/h_1 when compared with the theoretical curve indicate the accuracy of the measurements. On the other hand, the plots of H measured 4 feet downstream against h_0/h_1 indicate the reliability of the theory advanced herein. In the case of the initial strengths an error of measurement seems to exist which gives wave strengths consistently higher than the theoretical values. If the results are corrected for this consistent error, excellent agreement between theory and experiment is obtained.

CONCLUSIONS

A theory has been presented for the attenuation due to viscous boundary layer of shock waves traveling in closed passages. Because of the similar nature of the problem, the theory was also developed

for the attenuation of a hydraulic jump moving in a channel. The theory in its two forms was then checked experimentally.

Experimental results obtained by two greatly different methods indicate that the theory is sufficiently accurate to allow analysis of the effects of viscosity in shock tubes and intermittent-flow apparatus.

The assumption that the boundary layer in the intermittent-flow processes described depends only on time seems to be justified.

Experiments made in the shock tube show that, in the relatively small tube used, the pressure ratio across the shock wave dropped only some 12 to 15 percent in a distance of 11 feet. Thus, since the parameter controlling the attenuation is increased in proportion to length and to the inverse square of the radius, in the passages of a normal intermittent-flow apparatus the effect of viscosity at the walls causes little change in the strength of the waves moving in the passages.

Langley Aeronautical Laboratory
National Advisory Committee for Aeronautics
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APPENDIX

DERIVATION OF FORMULAS

Derivation of Formula for $m/\pi\rho_1 a_1 R^2$

The equations characterizing a moving plane shock may be found by changing the frame of reference so that the shock appears to be stationary. The velocities in this frame of reference are denoted by primes; the other quantities remain unchanged in the two frames of reference. The equations of a stationary normal shock are well-known. (See reference 5.) Thus, if the ratio of specific heats γ is taken as 1.4, the pressure ratio across the shock is given by

$$\frac{p_2}{p_1} = P = \frac{7M_1^2 - 1}{6} \tag{15}$$

and the density ratio by

$$\frac{\rho_2}{\rho_1} = \frac{6M_1^2}{M_1^2 + 5} \tag{16}$$

where

$$M_1 = \frac{u_1'}{a_1} \tag{17}$$

Further, the continuity equation is

$$u_1'\rho_1 = u_2'\rho_2 \tag{18}$$

In order to change the frame of reference back to the case of a shock moving into a still gas, the relations

$$\left.\begin{aligned} u_1' &= u_s \\ u_2' &= u_s - u_2 \end{aligned}\right\} \tag{19}$$

are used. By substituting values from equations (15) to (19) in the expression for mass flow

$$m = \pi\rho_2 u_2 R^2$$

the following relation, equation (5) in the text, is obtained:

$$\frac{m}{\pi\rho_1 a_1 R^2} = \frac{5}{\sqrt{7}}\sqrt{6P + 1}\,\frac{P - 1}{P + 6}$$

Derivation of Formula for $\nu_2 t/R^2$

If the gas law is written

$$P = \frac{p_2}{p_1} = \frac{\rho_2}{\rho_1}\,\frac{T_2}{T_1}$$

and if viscosity is assumed to vary linearly with temperature

$$\frac{\mu_2}{\mu_1} = \frac{T_2}{T_1}$$

the kinematic viscosity can be written

$$\nu_2 = \frac{\mu_2}{\rho_2} = \frac{\mu_1}{\rho_1}\,P\left(\frac{\rho_1}{\rho_2}\right)^2 = \nu_1 P\left(\frac{\rho_1}{\rho_2}\right)^2 \qquad (20)$$

Writing $t = \dfrac{x}{u_s}$ and using equations (15) to (20) result in

$$\frac{\nu_2 t}{R^2} = \frac{\sqrt{7}(P + 6)^2 P}{(6P + 1)^{2.5}}\,\frac{\nu_1 x}{a_1 R^2}$$

which is equation (10) in the text.

Derivation of Formula for p_0/p_1

Reference 6 shows that, on the two sides of an expansion wave, the relation

$$u_0 + \frac{2}{\gamma - 1} a_0 = u_2 + \frac{2}{\gamma - 1} a_2 \tag{21}$$

holds. In this case, $u_0 = 0$ and $a_0 = a_1$ since the original temperature on the two sides of the diaphragm is the same. If the speed of sound is taken as $a = \sqrt{\frac{\gamma p}{\rho}}$, the isentropic relation as $\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma}$, γ as 1.4 in equation (21), and by use of equations (15) to (19), the following expression is found for p_0/p_1:

$$\frac{p_0}{p_1} = P\left[1 - \frac{P - 1}{\sqrt{7(6P + 1)}}\right]^{-7}$$

Derivation of Formula for p_3/p_1

In order to determine the pressure in the end of the tube after the wave is reflected, a new frame of reference is taken so that the reflected wave appears to be stationary. This change is accomplished by setting

$$\left.\begin{aligned} u_1' &= u_{s_3} + u_2 \\ u_2' &= u_{s_3} \end{aligned}\right\} \tag{22}$$

where u_{s_3} is the velocity of the reflected shock.

Equations (15) to (19) can be solved for u_2 in terms of P, but equations (15) to (18) also apply to the reflected shock (after suitable changes in the subscripts) and, together with equation (22), can be solved for u_2 in terms of p_3/p_2. Equating these two expressions for u_2 and simplifying result in the expression

$$\frac{p_3}{p_2} = \frac{8P - 1}{P + 6}$$

Hence,

$$\frac{p_3}{p_1} = \frac{P(8P - 1)}{P + 6}$$

Equations for Hydraulic Jump

If a frame of reference is taken so that the jump appears to be stationary, the equation of continuity can be written as (see reference 4)

$$u_1'h_1 = u_2'h_2 \tag{23}$$

and the momentum equation as

$$\frac{gh_1^2}{2} + u_1'^2h_1 = \frac{gh_2^2}{2} + u_2'^2h_2 \tag{24}$$

By substituting values from equations (19), (23), and (24) in the expression for mass flow

$$m = \rho w u_2 h_2$$

the following relation, equation (12) in the text, is obtained:

$$\frac{m}{\rho w a_1 h_1} = \frac{1}{\sqrt{2}}(H - 1)\sqrt{H(H + 1)}$$

Also, the following expression for $\nu t/h_2^2$, equation (14) in the text, is found:

$$\frac{\nu t}{h_2^2} = \frac{1}{H^2}\sqrt{\frac{2}{H(H + 1)}}\,\frac{\nu x}{a_1h_1^2}$$

Since the expansion wave is a continuous phenomenon, equation (21) may be used in the hydraulic case in which $\gamma = 2$. (See reference 4.) Thus, if the speed of a small wave is $a = \sqrt{gh}$ and if $u_0 = 0$ in equation (21), and by use of equations (19), (23), and (24), the following expression for h_0/h_1 is obtained:

$$\frac{h_0}{h_1} = \left[\sqrt{H} + (H - 1)\sqrt{\frac{H + 1}{8H}}\right]^2$$

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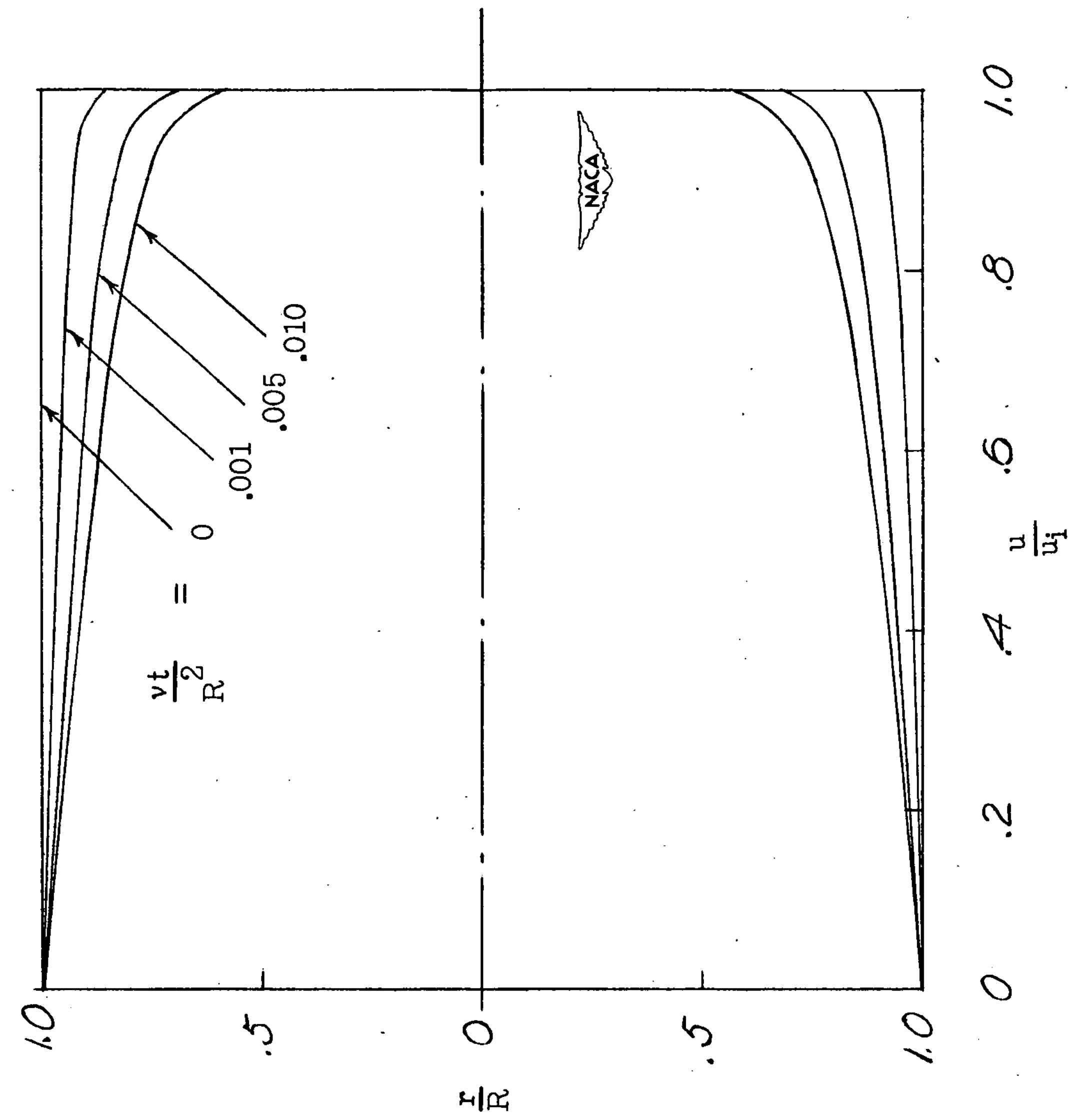

Figure 1.- Velocity distributions in a tube in which the fluid is suddenly set in motion at time $t = 0$.

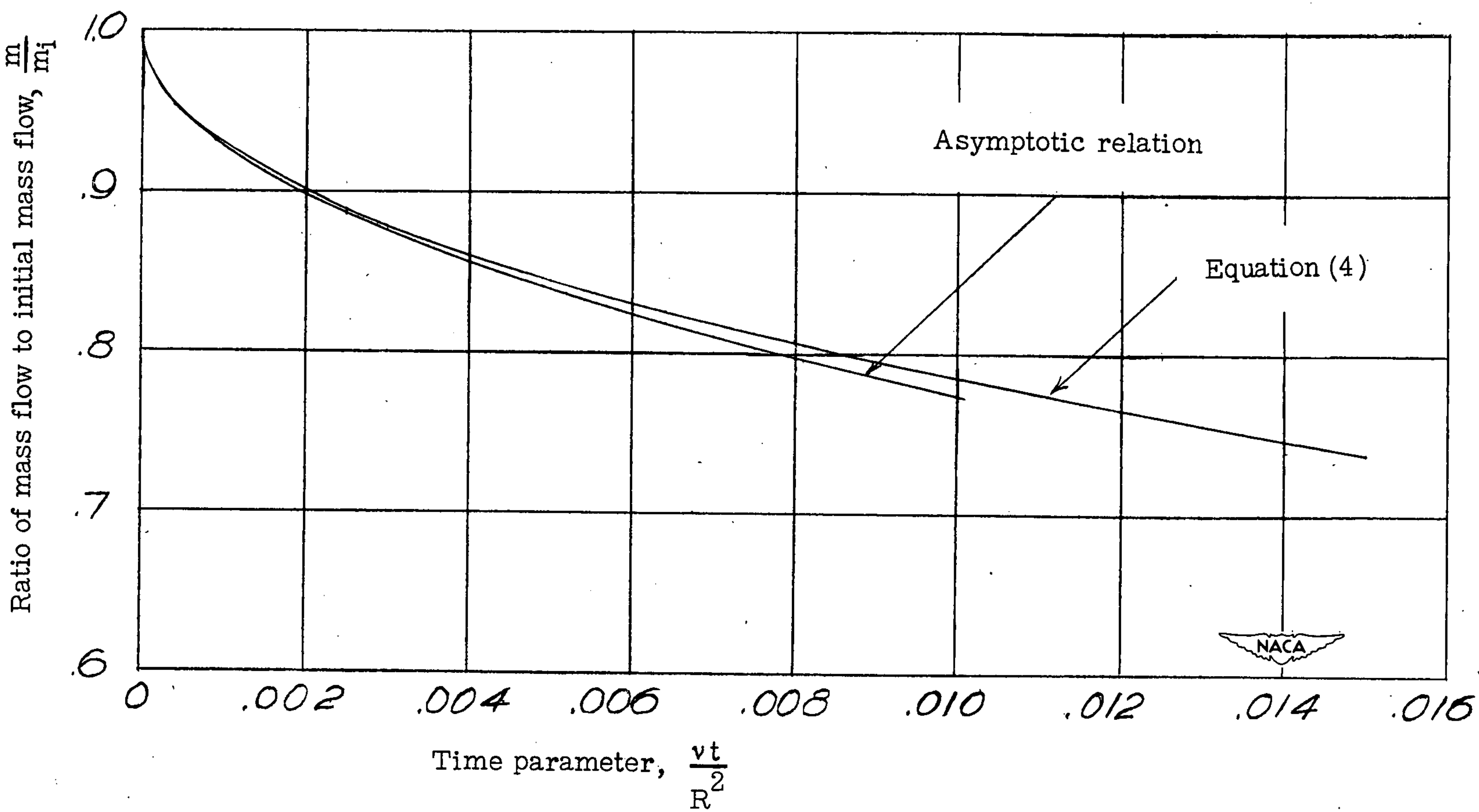

Figure 2.- Ratio of mass flow to initial mass flow in a shock tube given by equation (4) as a function of the time parameter $\frac{\nu t}{R^2}$ compared with the asymptotic relation for small values of $\frac{\nu t}{R^2}$.

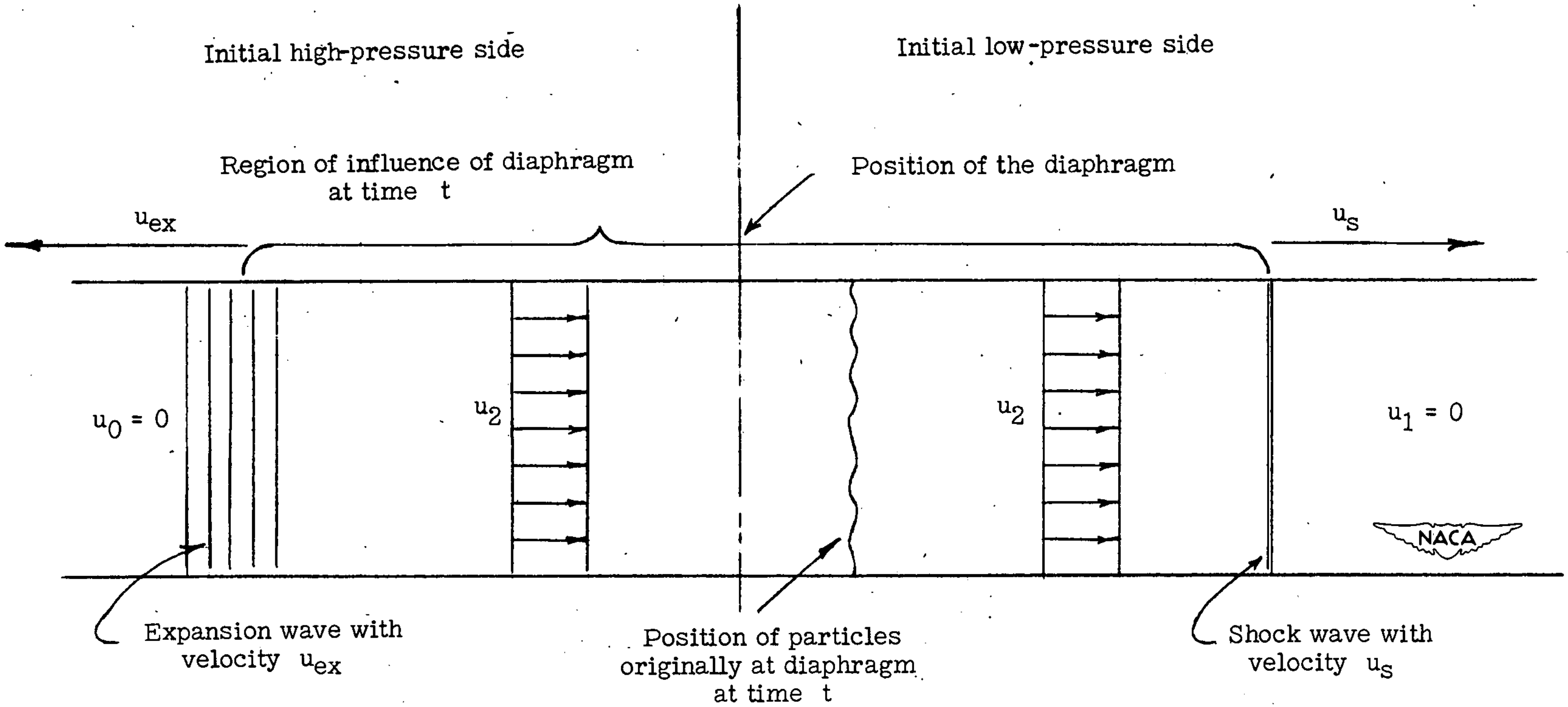

Figure 3.- Result, at time t, of bursting, at time $t = 0$, a diaphragm separating regions of different pressure in a tube.

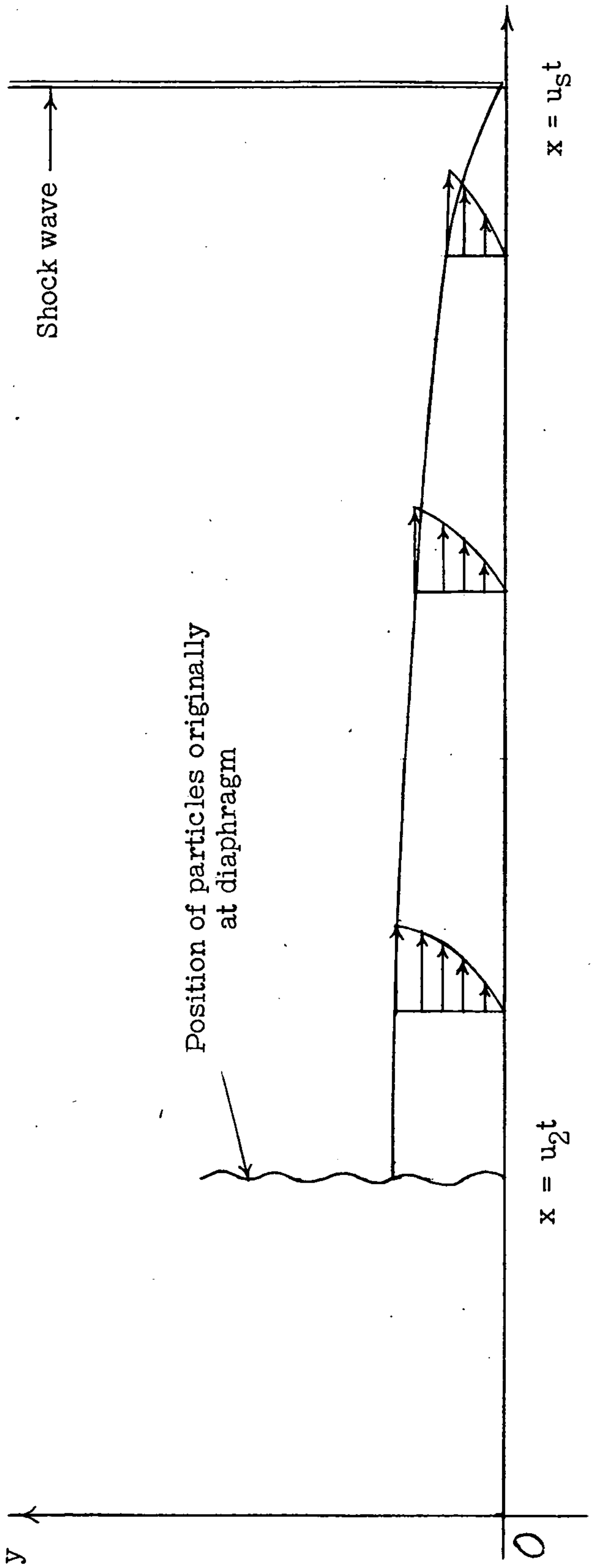

Figure 4.- Boundary layer on a wall behind a shock wave traveling into air at rest.

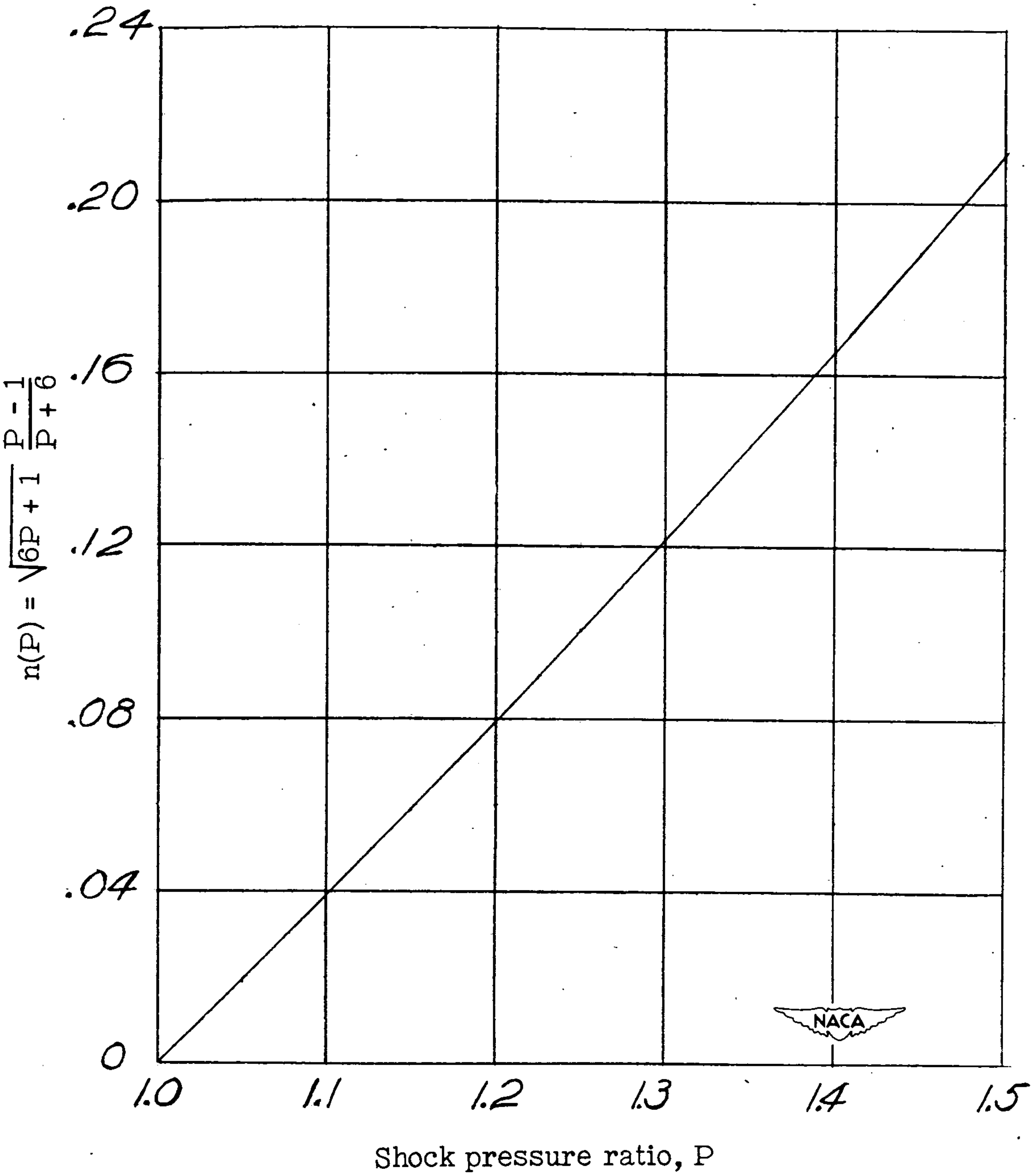

Figure 5.- Plot of the function n(P) from equation (8).

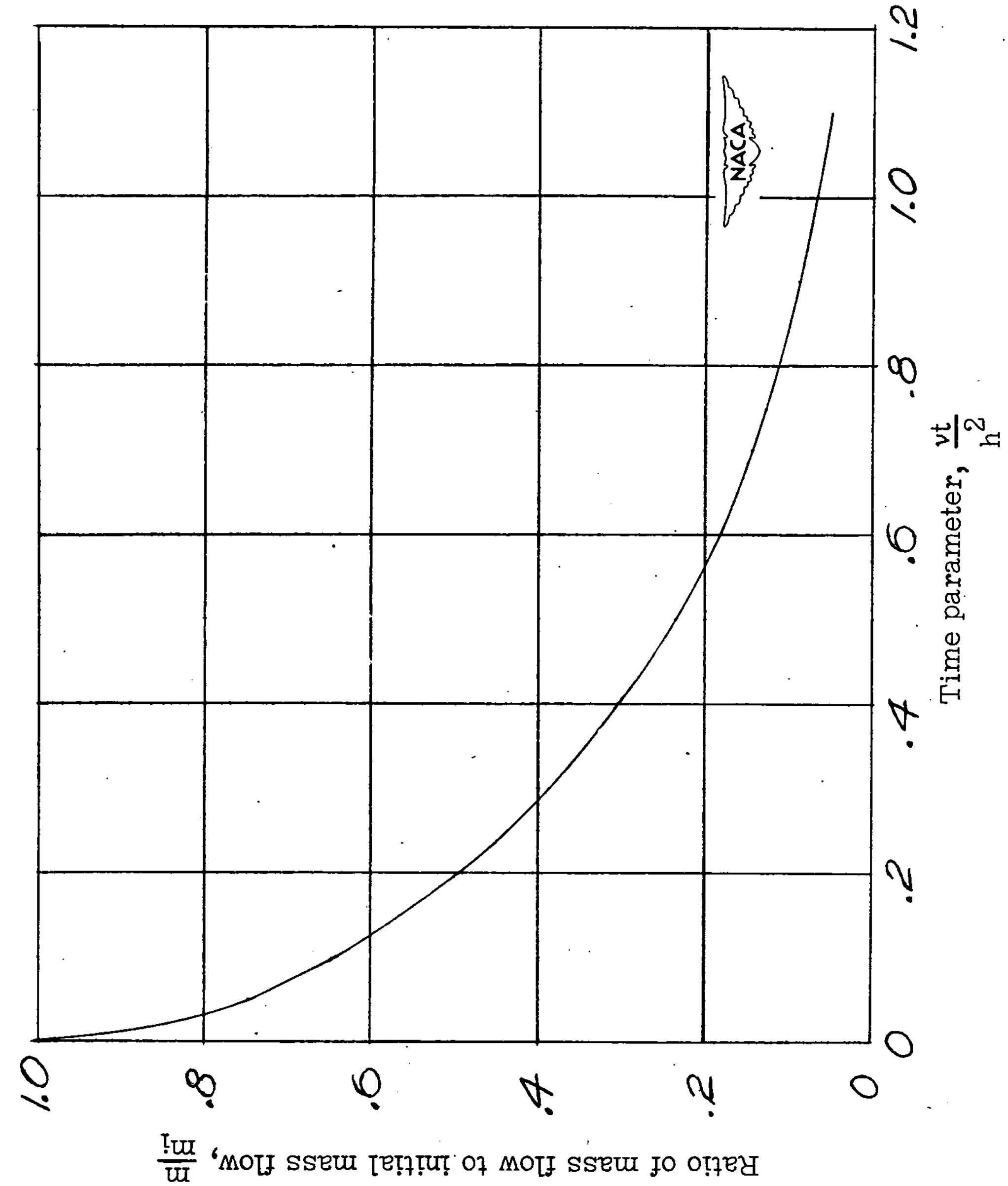

Figure 6.- Ratio of mass flow to initial mass flow in a shock wave given by equation (11) as a function of the time parameter $\frac{\nu t}{h^2}$.

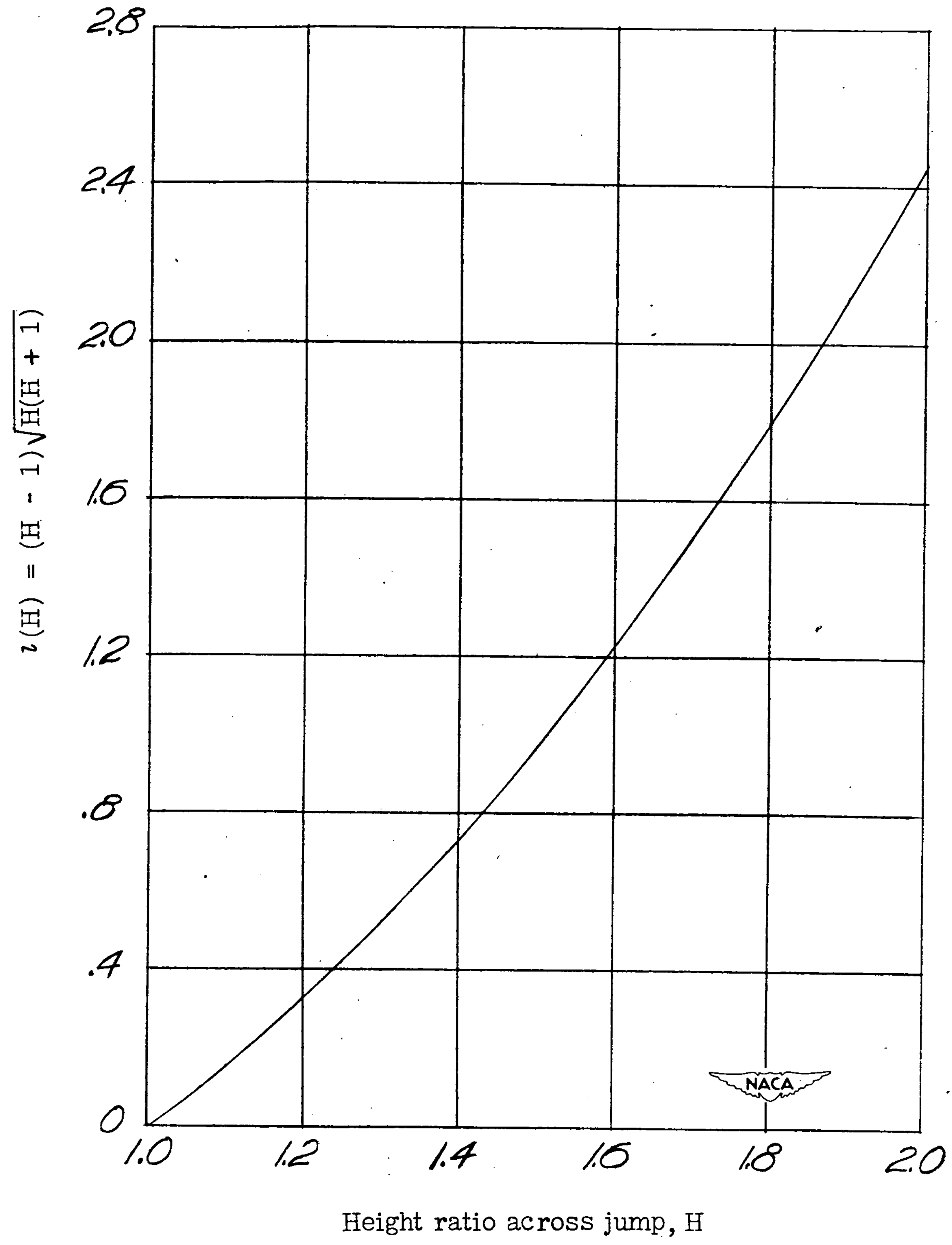

Figure 7.- Plot of the function $\imath(H)$ from equation (12).

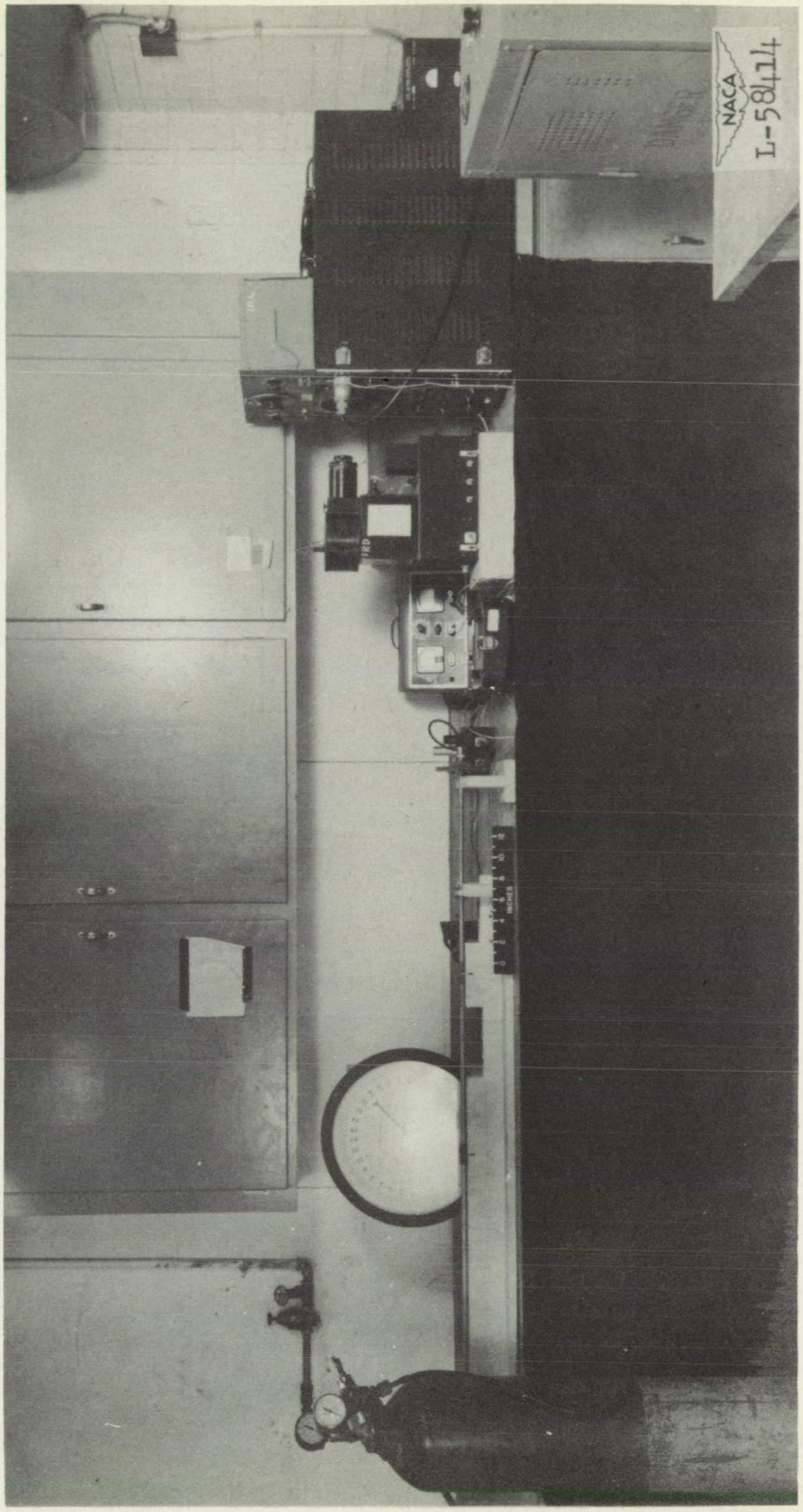

Figure 8.- Experimental apparatus.

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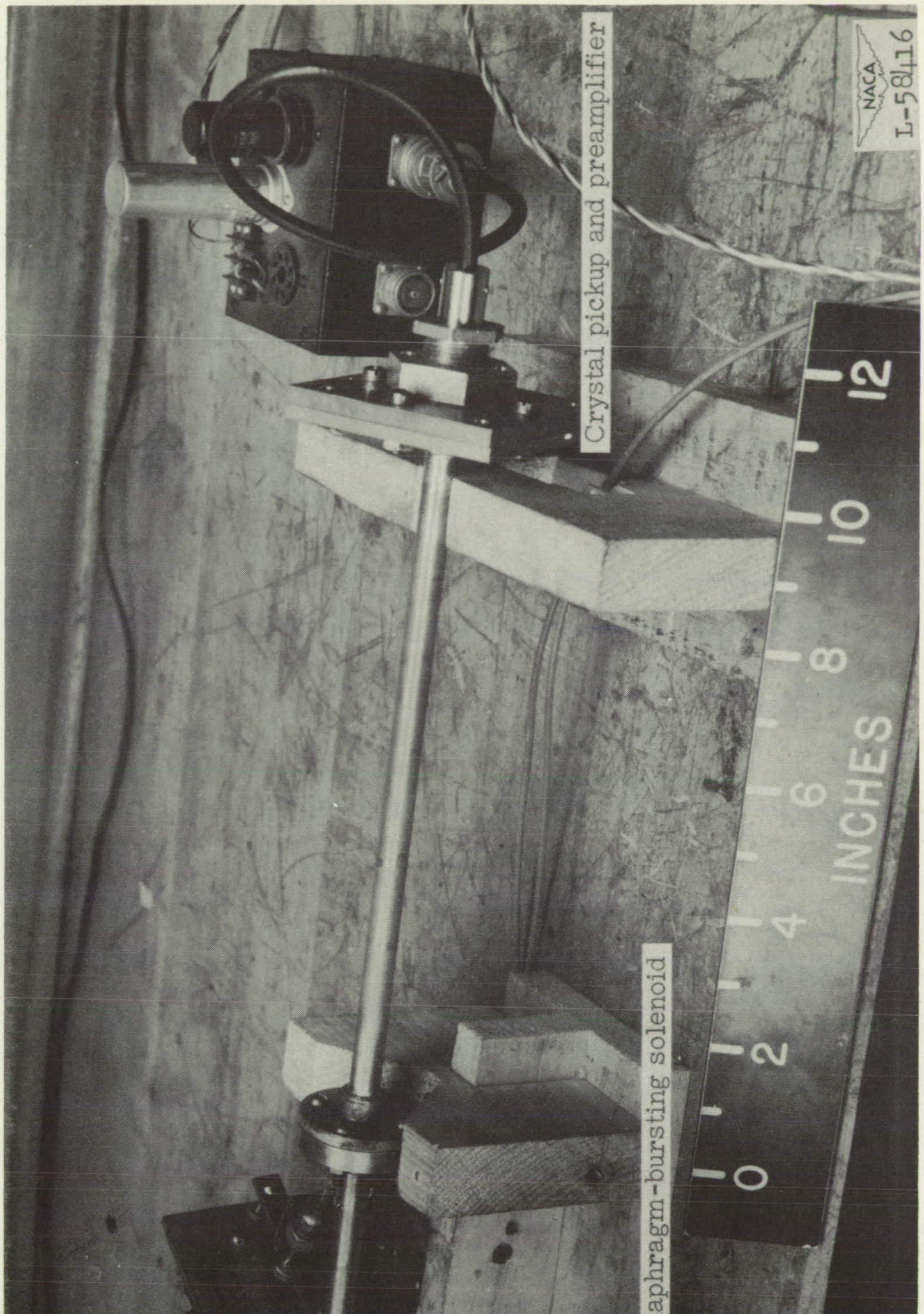

Figure 9.- Detail of short-shock-tube apparatus.

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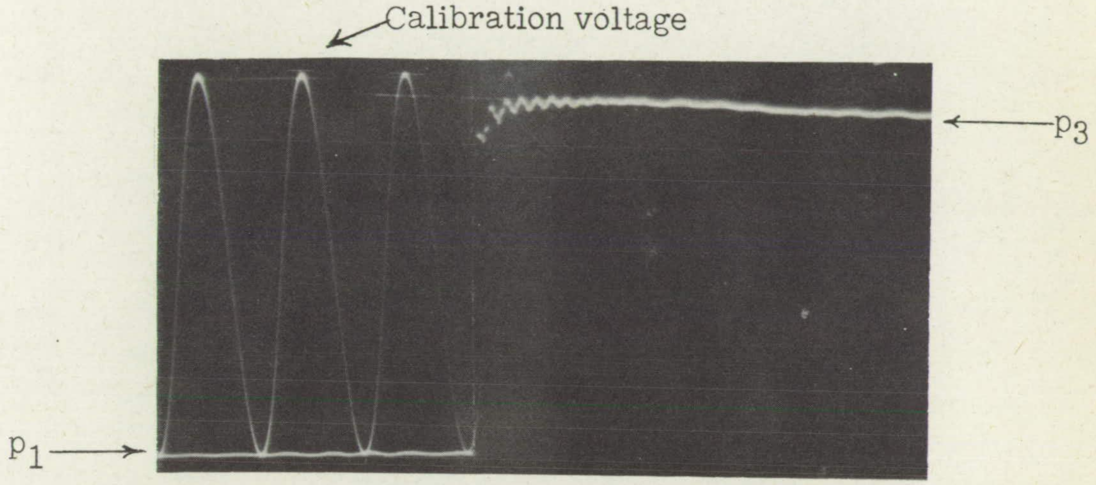

Figure 10.- Typical voltage record obtained when shock is reflected from end of tube.

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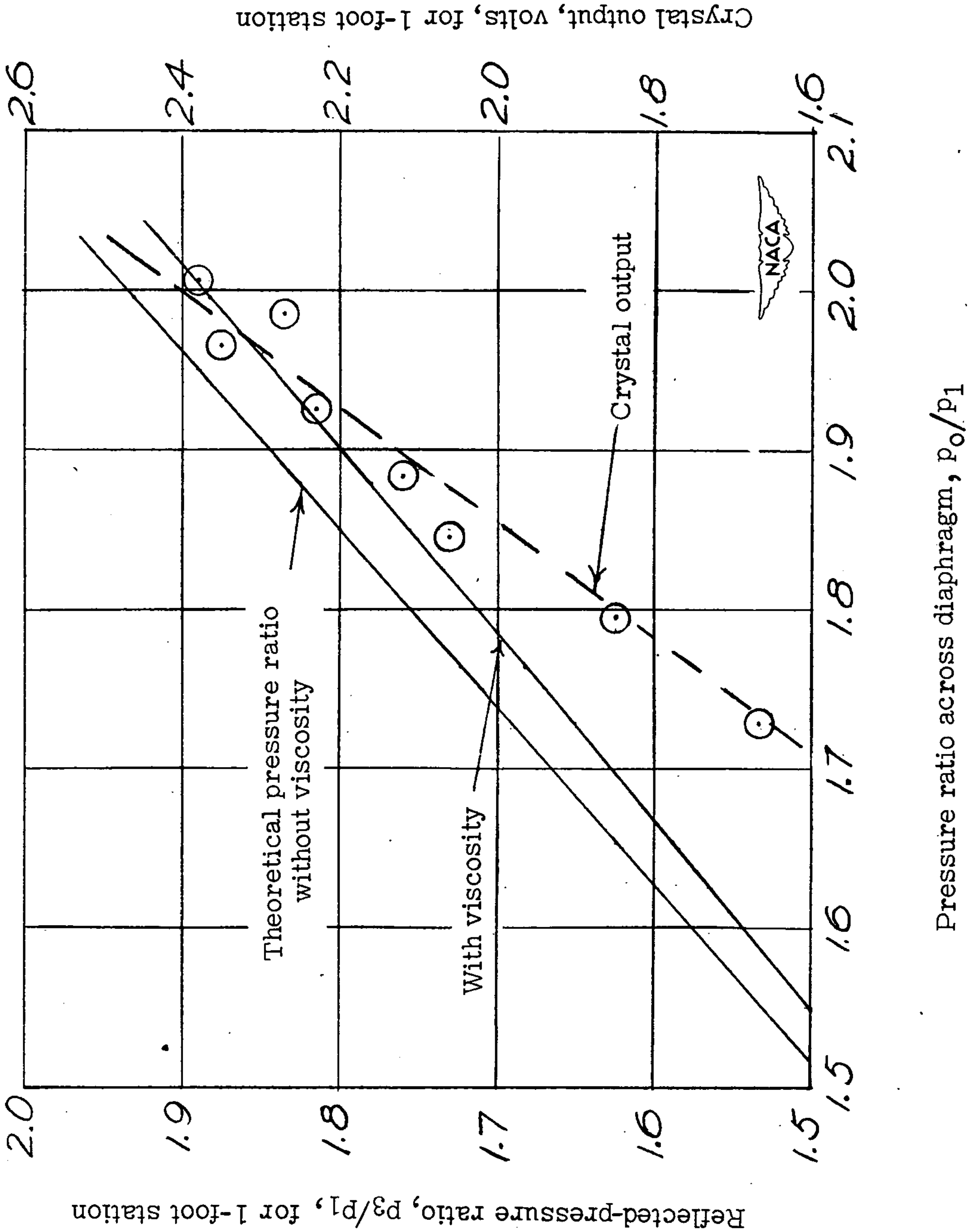

Figure 11.- Theoretical reflected-pressure ratios and measured crystal output for 1-foot tube.

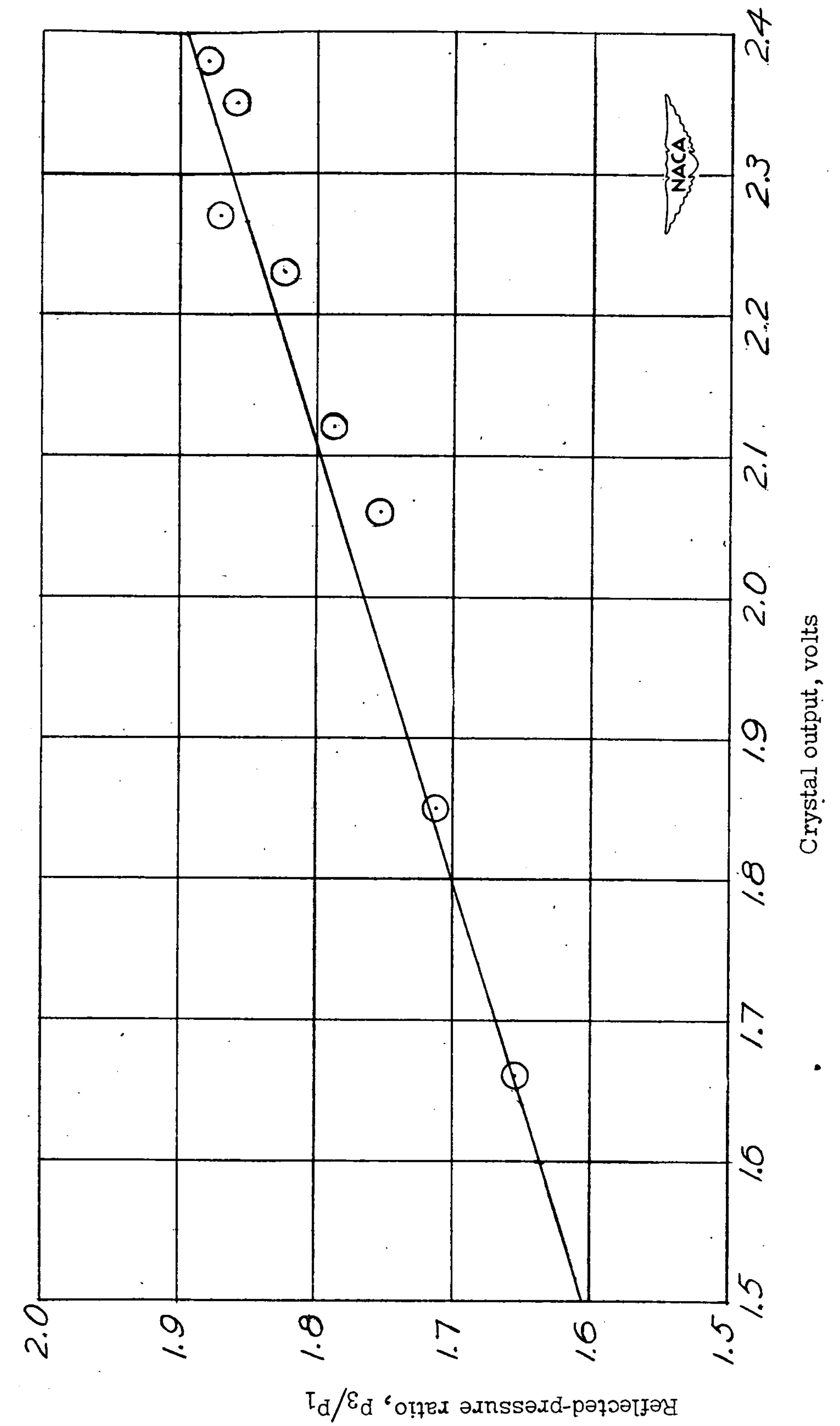

Figure 12.- Reflected-pressure ratio as a function of crystal voltage output.

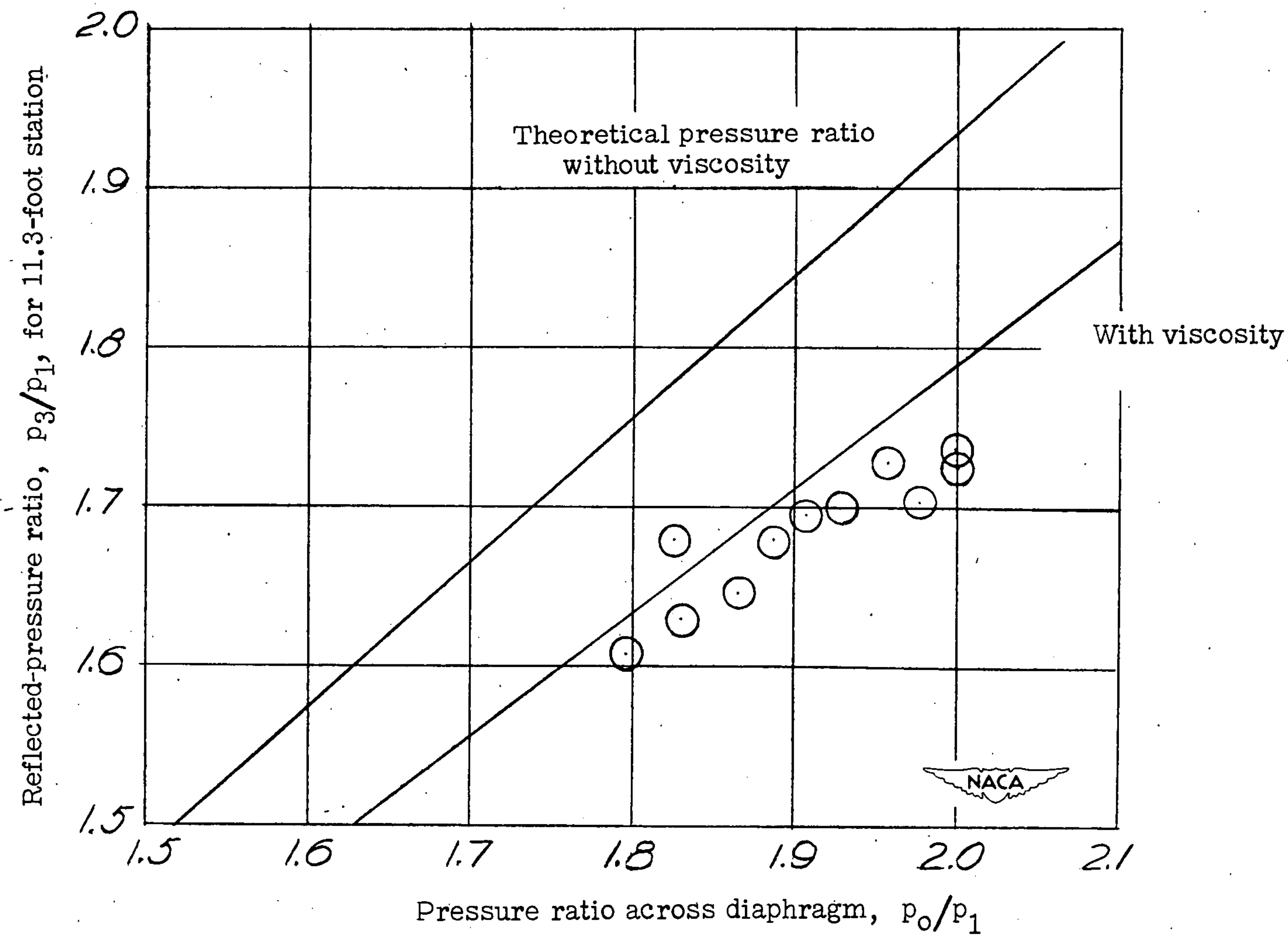

Figure 13.- Comparison of theoretical and measured reflected-pressure ratios for 11.3-foot tube.

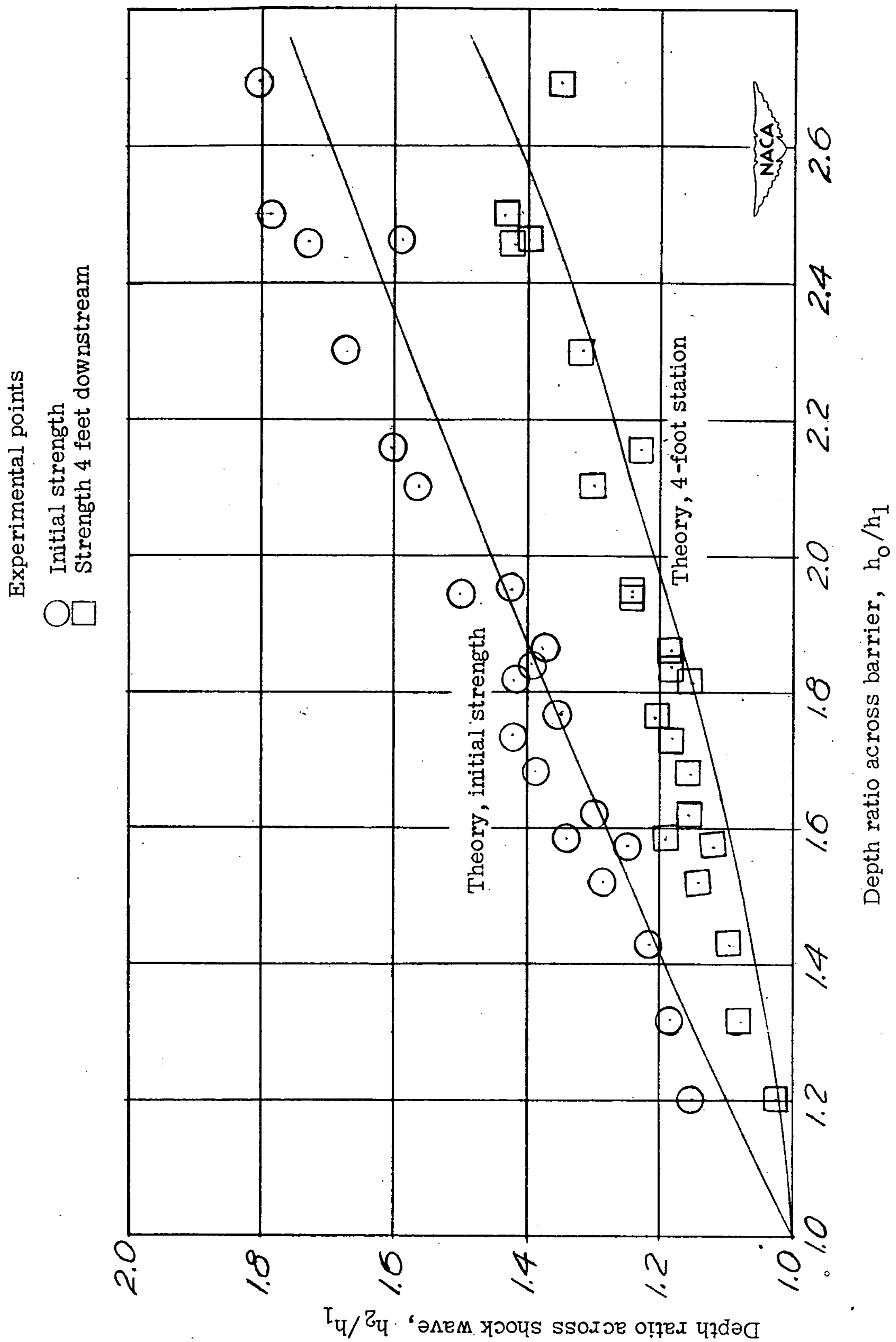

Figure 14.- Experimental results of hydraulic tests compared with theory.